The Rules of Sum and Product

The **Rule of Sum** and **Rule of Product** are used to decompose difficult counting problems into simple problems.

The Rule of Sum − If a sequence of tasks $T_1, T_2, \ldots, T_m$ can be done in $w_1, w_2, \ldots w_m$ respectively (the condition is that no tasks can be performed simultaneously), then the number of ways to do one of these tasks is $w_1 + w_2 + \cdots + w_m$. If we consider two tasks A and B which are disjoint (i.e. $A \cap B = \emptyset$), then mathematically $|A \cup B| = |A| + |B|$

The Rule of Product − If a sequence of tasks $T_1, T_2, \ldots, T_m$ can be done in $w_1, w_2, \ldots w_m$ ways respectively and every task arrives after the occurrence of the previous task, then there are $w_1 \times w_2 \times \cdots \times w_m$ ways to perform the tasks. Mathematically, if a task B arrives after a task A, then $|A \times B| = |A| \times |B|$

Example

Question − A boy lives at X and wants to go to School at Z. From his home X he has to first reach Y and then Y to Z. He may go X to Y by either 3 bus routes or 2 train routes. From there, he can either choose 4 bus routes or 5 train routes to reach Z. How many ways are there to go from X to Z?

Solution − From X to Y, he can go in $3+2=5$ ways (Rule of Sum). Thereafter, he can go Y to Z in $4+5=9$ ways (Rule of Sum). Hence from X to Z he can go in $5 \times 9 = 45$ ways (Rule of Product).

Permutations

A **permutation** is an arrangement of some elements in which order matters. In other words a Permutation is an ordered Combination of elements.

Examples

- From a set S ={x, y, z} by taking two at a time, all permutations are − xy,yx,xz,zx,yz,zy.
- We have to form a permutation of three digit numbers from a set of numbers S={1,2,3}. Different three digit numbers will be formed when we arrange the digits. The permutation will be = 123, 132, 213, 231, 312, 321

Number of Permutations

The number of permutations of 'n' different things taken 'r' at a time is denoted by nPr

$$nPr=n! / (n-r)!$$

where $n!=1.2.3.\ldots(n-1).n$

Proof − Let there be 'n' different elements.

There are n number of ways to fill up the first place. After filling the first place (n-1) number of elements is left. Hence, there are (n-1) ways to fill up the second place. After filling the first and second place, (n-2) number of elements is left. Hence, there are (n-2) ways to fill up the third place. We can now generalize the number of ways to fill up r-th place as $[n - (r-1)] = n-r+1$

So, the total no. of ways to fill up from first place up to r-th-place

$nPr=n(n-1)(n-2).....(n-r+1)$

$=[n(n-1)(n-2)...(n-r+1)][(n-r)(n-r-1)\ldots3.2.1]/[(n-r)(n-r-1)\ldots3.2.1]$

Hence,

$nPr=n!/(n-r)!$

Some important formulas of permutation

- If there are *n* elements of which a1 are alike of some kind, a2 are alike of another kind; a3 are alike of third kind and so on and ar are of rth kind, where (a1+a2+...ar)=n.
  Then, number of permutations of these n objects is = $n!/[(a1!(a2!)\ldots(ar!)]$.
- Number of permutations of n distinct elements taking n elements at a time = $nPn=n!$

- The number of permutations of n dissimilar elements taking r elements at a time, when x particular things always occupy definite places = $n-xpr-x$
- The number of permutations of n dissimilar elements when r specified things always come together is − $r!(n-r+1)!$
- The number of permutations of n dissimilar elements when r specified things never come together is − $n!–[r!(n-r+1)!]$
- The number of circular permutations of n different elements taken x elements at time = $^{n}p_{x}/x$
- The number of circular permutations of n different things = $^{n}p_{n}/n$

Some Problems

Problem 1 − From a bunch of 6 different cards, how many ways we can permute it?

Solution − As we are taking 6 cards at a time from a deck of 6 cards, the permutation will be $^{6}P6=6!=720$

Problem 2 − In how many ways can the letters of the word 'READER' be arranged?

Solution − There are 6 letters word (2 E, 1 A, 1D and 2R.) in the word 'READER'.

The permutation will be $=6!/[(2!)(1!)(1!)(2!)]=180$

Problem 3 − In how ways can the letters of the word 'ORANGE' be arranged so that the consonants occupy only the even positions?

Solution − There are 3 vowels and 3 consonants in the word 'ORANGE'. Number of ways of arranging the consonants among themselves $=^{3}P3=3!=6$. The remaining 3 vacant places will be filled up by 3 vowels in $^{3}P3=3!=6$ ways. Hence, the total number of permutation is $6×6=36$

Combinations

A **combination** is selection of some given elements in which order does not matter.

The number of all combinations of n things, taken r at a time is −

$$^{n}Cr=n! / r!(n-r)!$$

Problem 1

Find the number of subsets of the set {1,2,3,4,5,6}{1,2,3,4,5,6} having 3 elements.

Solution

The cardinality of the set is 6 and we have to choose 3 elements from the set. Here, the ordering does not matter. Hence, the number of subsets will be 6C3=20.

Problem 2

There are 6 men and 5 women in a room. In how many ways we can choose 3 men and 2 women from the room?

Solution

The number of ways to choose 3 men from 6 men is $^{6}C_{3}$and the number of ways to choose 2 women from 5 women is $^{5}C_{2}$

Hence, the total number of ways is − $^{6}C_{3} \times {}^{5}C_{2}=20 \times 10=200$

Problem 3

How many ways can you choose 3 distinct groups of 3 students from total 9 students?

Solution

Let us number the groups as 1, 2 and 3

For choosing 3 students for 1st group, the number of ways − $^{9}C_{3}$

The number of ways for choosing 3 students for 2nd group after choosing 1st group − $^{6}C_{3}$

The number of ways for choosing 3 students for 3rd group after choosing 1st and 2nd group − $^{3}C_{3}$

Hence, the total number of ways $= {}^{9}C_{3} \times {}^{6}C_{3} \times {}^{3}C_{3} = 84 \times 20 \times 1 = 1680$

Pascal's Identity

Pascal's identity, first derived by Blaise Pascal in 17th century, states that the number of ways to choose k elements from n elements is equal to the summation of number of ways to choose (k-1) elements from (n-1) elements and the number of ways to choose elements from n-1 elements.

Mathematically, for any positive integers k and n: $^{n}Ck = {}^{n-1}Ck-1 + {}^{n-1}Ck$

Proof −

$$n-1Ck-1+n-1Ck$$

$$=(n-1)! / (k-1)!(n-k)!+(n-1)! / k!(n-k-1)!$$

$$=(n-1)!(k / k!(n-k)!+n-k / k!(n-k)!)$$

$$=(n-1)!n / k!(n-k)!$$

$$=n! / k!(n-k)!$$

$$=nCk$$

Pigeonhole Principle

In 1834, German mathematician, Peter Gustav Lejeune Dirichlet, stated a principle which he called the drawer principle. Now, it is known as the pigeonhole principle.

Pigeonhole Principle states that if there are fewer pigeon holes than total number of pigeons and each pigeon is put in a pigeon hole, then there must be at least one pigeon hole with more than one pigeon. If n pigeons are put into m pigeonholes where $n > m$, there's a hole with more than one pigeon.

Examples

- Ten men are in a room and they are taking part in handshakes. If each person shakes hands at least once and no man shakes the same man's hand more than once then two men took part in the same number of handshakes.
- There must be at least two people in a class of 30 whose names start with the same alphabet.

The Inclusion-Exclusion principle

The **Inclusion-exclusion principle** computes the cardinal number of the union of multiple non-disjoint sets. For two sets A and B, the principle states −

$|A \cup B| = |A| + |B| - |A \cap B|$

For three sets A, B and C, the principle states −

$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$

The generalized formula -

$|\cup ni=1Ai| = \sum 1 \le i<j<k \le n |Ai \cap Aj| + \sum 1 \le i<j<k \le n |Ai \cap Aj \cap Ak| - \cdots$

$+(-1)\backslash n-1 |A1 \cap \cdots \cap A2|$

Problem 1

How many integers from 1 to 50 are multiples of 2 or 3 but not both?

Solution

From 1 to 100, there are $50/2=25$ numbers which are multiples of 2.

There are $50/3=16$ numbers which are multiples of 3.

There are $50/6=8$ numbers which are multiples of both 2 and 3.

So, $|A|=25$, $|B|=16$ and $|A \cap B|=8$.

$|A \cup B|=|A|+|B|-|A \cap B|=25+16-8=33$

Problem 2

In a group of 50 students 24 like cold drinks and 36 like hot drinks and each student likes at least one of the two drinks. How many like both coffee and tea?

Solution

Let X be the set of students who like cold drinks and Y be the set of people who like hot drinks.

So, $|X \cup Y|=50$, $|X|=24$, $|Y|=36$

$|X \cap Y|=|X|+|Y|-|X \cup Y|=24+36-50=60-50=10$

Hence, there are 10 students who like both tea and coffee.

Principle of Inclusion and Exclusion (PIE)

The **principle of inclusion and exclusion (PIE)** is a counting technique that computes the number of elements that satisfy at least one of several properties while guaranteeing that elements satisfying more than one property are not counted twice.

An underlying idea behind PIE is that summing the number of elements that satisfy at least one of two categories and subtracting the overlap prevents double counting. For instance, the number of people that have at least one cat or at least one dog can be found by taking the number of people who own a cat, adding the number of people that have a dog, then subtracting the number of people who have both.

PIE is particularly useful in combinatorics and probability problem solving when it is necessary to devise a counting method that ensures an object is not counted twice.

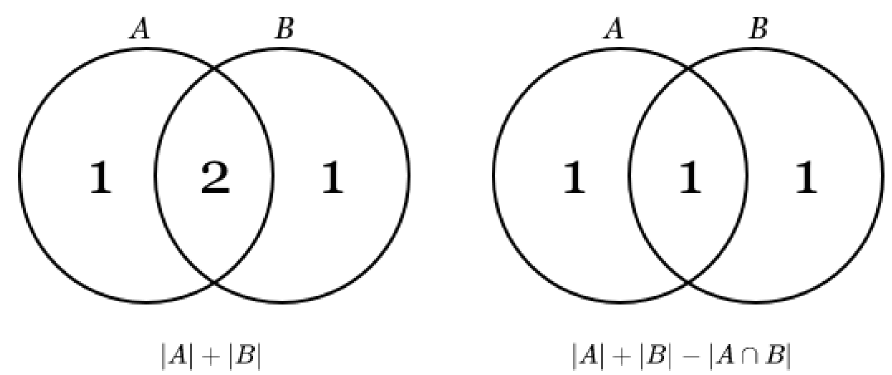
Two Sets

In the case of objects being separated into two (possibly disjoint) sets, the principle of inclusion and exclusion states

$|A \cup B| = |A| + |B| - |A \cap B|$,

where $|S|$ denotes the cardinality, or number of elements, of set S in set notation.

As a Venn diagram, PIE for two sets can be depicted easily:



The number in each subset represents how many times that subset is counted.
Note that by adding the elements in sets A and B, their intersection is counted twice.

Example:

How many integers from 1 to 100 are multiples of 2 or 3?

Let A be the set of integers from 1 to 100 that are multiples of 2, then $|A|=50$.
Let B be the set of integers from 1 to 100 that are multiples of 3, then $|B|=33$.
Now, $A \cap B$ is the set of integers from 1 to 100 that are multiples of both 2 and 3, and hence are multiples of 6, implying $A \cap B|=16$.

Hence, by PIE, $A \cup B|=|A|+|B|-|A \cap B|=50+33-16=67$.

Mathematical Induction

Definition

Mathematical Induction is a mathematical technique which is used to prove a statement, a formula or a theorem is true for every natural number.

The technique involves two steps to prove a statement, as stated below −

Step 1(Base step) − It proves that a statement is true for the initial value.

Step 2(Inductive step) − It proves that if the statement is true for the n^{th} iteration (or number *n*), then it is also true for $(n+1)^{th}$ iteration (or number *n+1*).

Problem 2

$1+3+5+...+(2n-1)=n^2$ for $n=1,2,\ldots$

Solution

Step 1 − For $n=1, 1=1^2$, Hence, step 1 is satisfied.

Step 2 − Let us assume the statement is true for $n=k$.

Hence, $1+3+5+\cdots+(2k-1)=k^2$ is true (It is an assumption)

We have to prove that $1+3+5+...+(2(k+1)-1)=(k+1)^2$ also holds

$1+3+5+\cdots+(2(k+1)-1)$

$=1+3+5+\cdots+(2k+2-1)$

$=1+3+5+\cdots+(2k+1)$

$=1+3+5+\cdots+(2k-1)+(2k+1)$

$=k2+(2k+1)$

$=(k+1)^2$

So, $1+3+5+\cdots+(2(k+1)-1)=(k+1)^2$ hold which satisfies the step 2.

Hence, $1+3+5+\cdots+(2n-1)=n^2$ is proved.

Problem 3

Prove that $(ab)^n = a^n b^n$ is true for every natural number n

Solution

Step 1 − For n=1,$(ab)^1 = a^1 b^1 = ab$, Hence, step 1 is satisfied.

Step 2 − Let us assume the statement is true for n=k, Hence, $(ab)^k = a^k b^k$ is true (It is an assumption).

We have to prove that $(ab)^{k+1} = a^k + 1b^{k+1}$ also hold

Given, $(ab)^k = a^k b^k$

Or, $(ab)^k (ab) = (a^k b^k)(ab)$ [Multiplying both side by 'ab']

Or, $(ab)^{k+1} = (aa^k)(bb^k)$

Or, $(ab)^{k+1} = (a^{k+1} b^{k+1})$

Hence, step 2 is proved.

So, $(ab)^n = a^n b^n$ is true for every natural number n.

<u>Bayes' theorem</u>

In probability theory and statistics, **Bayes' theorem** (alternatively **Bayes' law** or **Bayes' rule**), named after Thomas Bayes, describes the probability of an event, based on prior knowledge of conditions that might be related to the event.

What is Bayes' Theorem?

Bayes theorem is also known as the Bayes Rule or Bayes Law. It is used to determine the conditional probability of event A when event B has already happened. The general statement of Bayes' theorem is "The conditional probability of an event A, given the occurrence of another event B, is equal to the product of the event of B, given A and the probability of A divided by the probability of event B." i.e.

$P(A|B) = P(B|A)P(A) / P(B)$

where,
P(A) *and* ***P(B)*** *are the probabilities of events A and B*
P(A|B) *is the probability of event A when event B happens*
P(B|A) *is the probability of event B when A happens*

Bayes Theorem Statement

Bayes' Theorem for n set of events is defined as,

Let E_1, E_2,…, E_n be a set of events associated with the sample space S, in which all the events E_1, E_2,…, E_n have a non-zero probability of occurrence. All the events E_1, E_2,…, E form a partition of S. Let A be an event from space S for which we have to find probability, then according to Bayes' theorem,

$P(E_i|A) = P(E_i)P(A|E_i) / \sum P(E_k)P(A|E_k)$

for $k = 1, 2, 3, \ldots, n$

$$P(A|B) = \frac{P(B|A)\,P(A)}{P(B)}$$

Bayes' theorem is stated mathematically as the following equation:[17]

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

where A and B are events and $P(B) \neq 0$.

- $P(A|B)$ is a conditional probability: the probability of event A occurring given that B is true. It is also called the posterior probability of A given B.
- $P(B|A)$ is also a conditional probability: the probability of event B occurring given that A is true. It can also be interpreted as the likelihood of A given a fixed B because $P(B|A) = L(A|B)$.
- $P(A)$ and $P(B)$ are the probabilities of observing A and B respectively without any given conditions; they are known as the prior probability and marginal probability.

Numerical Example of Bayes' Theorem

Example 1: A person has undertaken a job. The probabilities of completion of the job on time with and without rain are 0.44 and 0.95 respectively. If the probability that it will rain is 0.45, then determine the probability that the job will be completed on time.
Solution:
Let E_1 be the event that the mining job will be completed on time and E_2 be the event that it rains. We have,
P(A) = 0.45,

P(no rain) = P(B) = 1 − P(A) = 1 − 0.45 = 0.55

By multiplication law of probability,

$P(E_1)$ = 0.44
$P(E_2)$ = 0.95
Since, events A and B form partitions of the sample space S, by total probability theorem, we have

$P(E) = P(A)\ P(E_1) + P(B)\ P(E_2)$
= 0.45 × 0.44 + 0.55 × 0.95

= 0.198 + 0.5225 = 0.7205

So, the probability that the job will be completed on time is 0.7205.

Example 1

What is the probability of a patient having liver disease if they are alcoholic?

Here, “being an alcoholic” is the “test” (type of litmus test) for liver disease.

- A is the event i.e. “patient has liver disease””.

As per earlier records of the clinic, it states that 10% of the patient’s entering the clinic are suffering from liver disease.

Therefore, P(A)=0.10

- B is the litmus test that “Patient is an alcoholic”.

Earlier records of the clinic showed that 5% of the patients entering the clinic are alcoholic.

Therefore, P(B)=0.05

- Also, 7% out of the he patient’s that are diagnosed with liver disease, are alcoholics. This defines the B|A: probability of a patient being alcoholic, given that they have a liver disease is 7%.

As, per **Bayes theorem formula**,

$P(A|B) = (0.07 * 0.1)/0.05 = 0.14$

Therefore, for a patient being alcoholic, the chances of having a liver disease are 0.14 (14%).

<u>Example2</u>

- Dangerous fires are rare (1%)
- But smoke is fairly common (10%) due to barbecues,
- And 90% of dangerous fires make smoke

What is the probability of dangerous Fire when there is Smoke?

Calculation

$P(Fire|Smoke) = P(Fire)\ P(Smoke|Fire)/P(Smoke)$

$= 1\% \times 90\%/10\%$

$= 9\%$